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Relativistic Feynman-Type Integrals

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It is shown that within the framework of the Kershaw stochastic model generalized by the author to the relativistic case a Feynman-type process may be constructed which can formally be understood as a diffusion phenomenon in Euclidean space. This makes it possible to introduce a real probability measure in the scheme of quantum mechanics proposed by Feynman.

1. INTRODUCTION

In the scheme of quantum mechanics proposed by Feynman (1948) an analogy of the probability measure is used which allows one to describe the behavior of quantum mechanical particles. Kac (1957) noticed that this measure is a complex quantity. This complex measure in the Feynman path integral corresponds to the presence of a factor i in the exponential for the Wiener measure (see Nash, 1978).

The presence of the i in the exponential causes uncontrollable oscillations in the path integral. This makes difficult the understanding of the Feynman path integral as a well-defined mathematical object. Despite this drawback it is a matter of history that the path integral is an extremely important contribution to quantum theory.

In the present paper we make an attempt to generalize the definition of the Feynman path integral to the relativistic case within the framework of the theory of stochastic processes started by Kershaw (1964) and Nelson (1966) (see the review by Moore, 1979). Conditionally, it is considered that there exist two approaches to the generalization of the Feynman scheme to the relativistic case. One of them was started by Feynman (1950, 1951) and is based on a formal generalization of the notion of path in the four-dimensional space and time. In this case the path is defined by four functions $x^{\mu}(s)$, where s is some invariant parameter (proper time). In the paper of Miura (1979) relativistic path integrals are investigated by using Weyl's (1929, 1950) gauge theory.

Supporters of the second approach assume that a strict definition of relativistic integrals is possible, if the Feynman process can be considered from the point of view of the relativistic-invariant description of Markov diffusion processes (see, for example Guerra and Ruggiero, 1978). The difficulty of this approach is as follows: Until now a satisfactory scheme of the relativistic-invariant description of diffusion processes is absent. Many authors, in particular, Guerra and Ruggiero (1978), Lehr and Park (1977), Vigier (1979), and Namsrai (1980) have performed studies in this direction.

In the previous paper (Namsrai, 1980) we have considered a method for an extension of the stochastic model of diffusion processes to the relativistic case. The basic hypothesis was as follows:

(i) The physical quantities are considered as functions of complex times $t+i\tau$ in the limit $\tau \to 0$.

(ii) It is assumed that the stochastic behavior of a particle takes place in the Euclidean space $(\hat{x}_i, \tau)$, but not in the Minkowski space $(\hat{x}_i, t)$.

Using the language of random fluctuations this means that the fluctuations appear in the Euclidean space $E_4(\hat{x}_i, \tau)$. The importance of the method of shift $x_0 \to x_0 + i\tau$ in the time variable in quantum field theory and quantum mechanics was noted by Alebastrov and Efimov (1974) and Davidson (1978), respectively.

We have constructed in this paper within the framework of this approach the Feynman process by using Smoluchowski-type equations. These equations allow one to obtain easily the Schrödinger, Klein–Gordon, and Dirac equations. The interaction in the Smoluchowski-type equations for fields φ is introduced by using Weyl's (1929, 1950) gauge theory (see also Miura, 1979).

In our model the Feynman process may formally be interpreted as a stochastic process in complex times with a real probability measure which occurs in the Euclidean space.

2. DIFFUSION PROCESS IN REAL TIME

In the language of motion of a stochastic particle the property of Markov process means that the character of displacement of a particle at given time does not depend on the property of previous displacements. Accordingly, the position probability density $\rho(x_i, t)$ must obey the Smoluchowski equation

$$\rho(x_i, t+\Delta t) = \int \rho(x_i - \delta x_i, t) P_0(x_i - \delta x_i, t;\ \delta x_i, \Delta t) d^3(\delta x) \qquad (2.1)$$

where P_0 is the conditional probability density that a particle at position $x_i - \delta x_i$ at time t will be displaced by δx_i during the interval Δt, thus reaching position x_i at time $t + \Delta t$. The simple form

$$P_0 = (4\pi \mathfrak{D}\, \Delta t)^{-3/2} \exp\left[-\frac{(\delta x_i)^2}{4\mathfrak{D}\, \Delta t} \right] \tag{2.2}$$

reduces to the diffusion equation for $\rho(x_i, t)$:

$$\frac{\partial \rho}{\partial t} = \mathfrak{D}\, \nabla^2 \rho \tag{2.3}$$

here $\mathfrak{D}$ is the diffusion coefficient.

Following our model (Namsrai, 1980) in the relativistic case we consider formally the motion of a particle suffering the random flights owing to stochasticity of the four-dimensional Euclidean space $E_4(\hat{x}_i, \tau)$. Then equation (2.1) acquires the following form:

$$\rho(x_\mu, u + \Delta u) = \int \rho(x_i - y_i, x_0 + iy_4, u) P_1(x_i - y_i, x_0 + iy_4, u; y_E, \Delta u)\, d^4 y_E, \tag{2.4}$$

where the variables $x_\mu = (x_0, x_i)$ are pseudo-Euclidean and P_1 can be chosen in the form

$$P_1 = (4\pi \mathfrak{D}\, \Delta u)^{-2} \exp\left\{ -\frac{y_E^{\,2}}{4\mathfrak{D}\, \Delta u} \right\} \tag{2.5}$$

here u is some invariant parameter (proper time) which may be interpreted as the fifth parameter introduced by Miura (1979). From (2.4) and (2.5) we have

$$\frac{\partial \rho}{\partial u} = \mathfrak{D} \Box \rho$$

$$\Box = -\frac{\partial^2}{\partial x_0^{\,2}} + \frac{\partial^2}{\partial x_i^{\,2}} \tag{2.6}$$

Notice that a more complicated form of the functions (measures) P_0 and P_1 make Fokker–Planck equations for ρ both in the nonrelativistic and relativistic cases, respectively, (see Namsrai, 1980).

3. "DIFFUSION PROCESS" IN COMPLEX TIMES

Now a basic postulate is that a field $\varphi(x_i, t)$ (probability amplitude) associated with a particle suffers a transformation and is defined as a diffusion process in complex time, whenever the particle displaces from point $x_i - \delta x_i$ at time t to position x_i at $t + \Delta t$. Then the corresponding Smoluchowski-type equation for the field $\varphi(x_i, t)$ becomes

$$\varphi(x_i, t - i\Delta t) = \int \varphi(x_i - \delta x_i, t) P_0(\delta x_i;\, \Delta t) d^3(\delta x) \tag{3.1}$$

where P_0 is given by (2.2). It is easily seen that from equation (3.1) we obtain the Schrödinger equation assuming $\mathfrak{D} = \hbar/(2m)$.

Substituting $\Delta t \to -i\Delta t$ into equation (3.1) gives the Feynman integral, and therefore in the nonrelativistic case our postulate does not make a new result in the method of the Feynman path integrals. An essential difference appears in the construction of the relativistic Feynman-type integrals by using the diffusion processes. We now pass to this question.

Roughly speaking, in the relativistic case the Feynman-type integrals for the probability amplitude are formally replaced by Smoluchowski-type equations at complex times. So, if the Feynman process $\varphi(x_\mu, u)$ is known at one value of u, its value at a slightly larger value $u - i\Delta u$ is given by

$$\varphi(x_\mu, u - i\Delta u) = \int \varphi(x_i - y_i, x_0 + iy_4, u) P_1(y_E, \Delta u) d^4 y_E \tag{3.2}$$

From expression (2.5) and equation (3.2) we obtain the Klein–Gordon equation in the parametric form

$$i\frac{\partial \varphi}{\partial u} = -\mathfrak{D}\Box\varphi \tag{3.3}$$

Sometimes, instead of equation (3.3) the following equation is considered:

$$i\frac{\partial \varphi}{\partial u} = -\mathfrak{D}(\Box - m^2)\varphi \tag{3.4}$$

which is obtained by using the measure

$$d\mu(y_E, \Delta u) = d^4 y_E (4\pi\mathfrak{D}\,\Delta u)^{-2} \exp\left(-m^2\mathfrak{D}\,\Delta u - \frac{y_E^2}{4\mathfrak{D}\,\Delta u}\right)$$

Formal formulas (3.1) and (3.2) will be interpreted as well-defined mathematical objects—some integral equations with a real probability measure of the Gaussian type. Here quantity $\mathfrak{D}$ is real always.

4. INTRODUCTION OF INTERACTIONS INTO THE SCHEME

As is clear, in the nonrelativistic case our formalism is equivalent to the Feynman integral if $\Delta t \to -i\Delta t$. Then due to Feynman we can write equation (3.1) in a potential field $U(x_i)$ by the following formula:

$$\varphi(x_i, t - i\Delta t) = \exp\left[-\frac{\Delta t}{2m\mathfrak{D}} U(x_i) \right] \cdot \int \varphi(x_i - \delta x_i, t) P_0(\delta x_i, \Delta t) d^3(\delta x)$$

From this we obtain the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial \varphi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2 \varphi + U(x_i)\varphi$$

if $\mathfrak{D} = \hbar/(2m)$.

In the relativistic case we introduce the interactions into our scheme within the framework of Weyl's gauge theory (see, Miura, 1979). Following this theory, the field takes the value $\varphi(x_\mu + dx_\mu, \Delta\mathcal{P})$ after the transport connected with the displacement of a particle from a world point $\mathcal{P}$ (coordinates x_μ) to a position $\mathcal{P}'$ (coordinates $x_\mu + dx_\mu$), and therefore the variation $\delta\varphi$ of φ made by this transport is given by

$$\delta\varphi = \varphi(x_\mu + dx_\mu, \Delta\mathcal{P}) - \varphi(x_\mu) = d\chi \cdot \varphi(x_\mu) \tag{4.1}$$

If $d\chi$ is the total differential of a coordinate function $ie\lambda(x_\mu)/(\hbar c)$, i.e.,

$$d\chi = i\frac{e}{\hbar c}\frac{\partial \lambda}{\partial x_\mu} dx_\mu$$

then (4.1) affects only an arbitrary phase of φ. Generally, $d\chi$ is given in the form

$$d\chi = -i\frac{e}{\hbar c} A_\mu dx_\mu \tag{4.2}$$

where A_μ is an electromagnetic potential. Assume $dx_\mu = x_\mu - x'_\mu$ and rewrite (4.1) in the following form:

$$\varphi(x_\mu, \Delta\mathcal{P}) = \exp\{d\chi\}\varphi(x'_\mu) = \exp\left(-\frac{ie}{\hbar c} A_\mu dx_\mu\right)\varphi(x'_\mu) \tag{4.3}$$

By using this formula the Feynman path integral for a Klein–Gordon particle may be defined in the form (Miura, 1979)

$$\varphi(x_\mu, u+\varepsilon) = \int \exp\left(\frac{i}{\hbar} S_0\right) \varphi(x_\mu, \Delta\mathcal{P}) d^4x' \prod_1^4 (\pm 2\pi i\hbar\varepsilon)^{-1/2}$$

where S_0 is the action of a free particle.

Because the displacement of variables x_i and x_0 in our case is of the Euclidean character, the corresponding formula (4.3) must be changed in the following manner:

$$\varphi(x_\mu, \Delta\mathcal{P}) = \exp\left(\frac{ie}{2m\mathcal{D}c} A_\mu{}^E y_\mu\right) \varphi(x_i - y_i, x_0 + iy_4) \tag{4.4}$$

where $A_\mu{}^E = (-iA_0, A_i)$ and $A_\mu{}^E y_\mu = A_4 y_4 + A_i y_i$. Then we obtain the value of φ at a space-time point after the transformation using averaging over φ shifted by all possible Euclidean displacements with a real probability measure $P_1(y_E, \Delta u) d^4 y_E$ of the Gaussian type and multiplied further by a weight function $\exp[(ie/2m\mathcal{D}c) A_\mu{}^E y_\mu]$ for an infinitesimal value of y_μ;

$$\varphi(x_\mu, u - i\Delta u) = \int \exp\left(\frac{ie}{2m\mathcal{D}c} A_\mu{}^E y_\mu\right) \varphi(x_i - y_i, x_0 + iy_4, u)\, d\mu(y_E, \Delta u) \tag{4.5}$$

where

$$d\mu(y_E, \Delta u) = P_1(y_E, \Delta u) d^4 y_E$$

From this we obtain the parametric Klein–Gordon equation in a external field

$$i\frac{\partial \varphi}{\partial u} = \mathcal{D}\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{2m\mathcal{D}c} A_\mu\right)^2 \varphi$$

if $\mathcal{D} = \hbar/(2m)$.

The generalization of our formalism to a Dirac particle does not represent any difficulty; for example, expressions (4.4) and (4.5) acquire the following form:

$$\psi(x_\mu, \Delta\mathcal{P}) = \exp\left[\frac{ie}{2} \frac{1}{2m\mathcal{D}c} (\hat{A}^E \gamma_\nu y_\nu + y_\nu \gamma_\nu \hat{A}'^E)\right] \psi(x_i - y_i, x_0 + iy_4) \tag{4.6}$$

and

$$\psi(x_\mu, u - i\Delta u) = \int \exp\left[\frac{ie}{2}\frac{1}{2m\mathfrak{D}c}(\hat{A}^E\gamma_\nu y_\nu + y_\nu\gamma_\nu\hat{A}'^E)\right]$$

$$\times\, \psi(x_i - y_i, x_0 + iy_4, u)\, d\mu(y_E, \Delta u) \tag{4.7}$$

where

$$A'^E_\mu = A_\mu{}^E(x_i - y_i, x_0 + iy_4) \qquad \text{and} \qquad \hat{A}^E = A_\mu{}^E\gamma_\mu$$

γ_μ are Dirac matrices.

After some calculations we have the Dirac equation in the parametric form

$$i\frac{\partial\psi}{\partial u} = \mathfrak{D}\left[\frac{\partial}{\partial x_\mu} - \frac{ie}{2m\mathfrak{D}c}A_\mu\right]^2\psi - \frac{e}{4c}\frac{i}{2m}\sigma_{\mu\nu}F_{\mu\nu}\psi,$$

here

$$\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \qquad F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

5. CONCLUSION

Equations (3.2), (4.5), and (4.7) we have obtained, generally speaking, have nothing to do with the Smoluchowski-type equations which describe the probability consequence, and therefore value $P_1(y_E, \Delta u)$ is not interpreted as the probability transition. On the contrary, these equations may be interpreted as a formal exposition of some mathematical objects obtained by integrating with the real probability measure $d\mu(y_E, \Delta u)$ of the Gaussian type. Feynman path integrals themselves are not obtained in our scheme owing to the fact that variables x_i and x_0 are shifted in a different way: $x_i \to x_i + y_i$ and $x_0 \to x_0 + iy_4$.

However, our method is interesting as representing one possibility for the relativistic generalization of Feynman-type integrals.

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